Kondrashov Compactness

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We show the following:

Theorem 1.1 (Kondrashov Compactness for $H^s(\mathbf{R}^n)$)**.** *Let $\mathcal{E}'(K)$ be the space of distributions whose support lies in the compact set K, and $t > s \in \mathbf{R}$. Then the inclusion*

$$H^t(\mathbf{R}^d) \cap \mathcal{E}'(K) \hookrightarrow H^s(\mathbf{R}^d)$$

is compact.

Set $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Let φ_ε be a standard system of mollifiers. Notice that for $v \in H^t(\mathbf{R}^d)$,

$$\begin{aligned} ||v * \varphi_\varepsilon||_{C^k(\mathbf{R}^d)} &\lesssim ||\langle \xi \rangle^k \hat{v} \hat{\varphi}_\varepsilon||_{L^1} \\ &\leq ||\hat{v} \langle \xi \rangle^t||_{L^2} ||\hat{\varphi}_\varepsilon \langle \xi \rangle^{k-t}||_{L^2} \\ &= C_\varepsilon ||v||_{H^t(\mathbf{R}^d)}, \end{aligned}$$

with $C_\varepsilon < \infty$ since $\hat{\varphi}_\varepsilon \in \mathcal{S}(\mathbf{R}^d)$. Also observe that $v * \varphi_\varepsilon$ is supported in $\operatorname{supp}(v) + B(0, \varepsilon)$.

It follows that if $u_n \in H^t(\mathbf{R}^d) \cap \mathcal{E}'(K)$ is uniformly bounded in H^t, then for every $\varepsilon > 0$, and $k > s$, $u_n * \varphi_\varepsilon$ are uniformly bounded together with all their derivatives. Furthermore, their supports lie in the compact set $L = K + \overline{B(0, \varepsilon)}$. From Arzela-Ascoli, it follows that for each $\varepsilon > 0$ there is a subsequence n_ℓ^ε for which

$$u_{n_k^\varepsilon} * \varphi_\varepsilon \to u_\varepsilon \in C^0(L)$$

in $C^0(L)$. However, the same is true of their derivatives. Since $K + B(0, \varepsilon)$ is open, we deduce that the convergence is actually in $C^k(L)$ (we lose a derivative since it provides the equicontinuity), and hence in $H^k(\mathbf{R}^d)$ as well, since all functions are compactly supported in L, and hence in $H^s(\mathbf{R}^d)$. Iterating this argument, we may thus pick a diagonal sequence n_k so that $u_{n_k} * \varphi_\varepsilon$ is convergent, and hence Cauchy in $H^s(\mathbf{R}^d)$ for $\varepsilon = 1, 1/2, 1/3, \ldots$.

Next, we show that if $v \in H^t(\mathbf{R}^d)$, then for all $\delta > 0$ and $\varepsilon > 0$ is small enough, then $||v - v * \varphi_\varepsilon||_{H^s(\mathbf{R}^d)} < \delta ||v||_{H^t(\mathbf{R}^d)}$. To prove this, notice that

$$\widehat{v - v * \varphi_\varepsilon} = \hat{v}(1 - \hat{\varphi}_\varepsilon),$$

and that $\hat{\varphi}_\varepsilon \to 1$ uniformly on compact sets. So

$$||v - v * \varphi_\varepsilon||_{H^s(\mathbf{R}^d)} = ||\hat{v}\langle\xi\rangle^t(1 - \hat{\varphi}_\varepsilon)\langle\xi\rangle^{s-t}||_{L^2(\mathbf{R}^d)}. \tag{1.1}$$

Since $s < t$, $\langle\xi\rangle^{s-t} \to 0$ as $\xi \to \infty$, in particular if R is large enough, $\langle\xi\rangle^{s-t} < \delta$ if $|\xi| > R$. On $|\xi| \leq R$, we may pick ε small enough so that $(1 - \varphi_\varepsilon) < \delta$. Since $\langle\xi\rangle^{s-t} \leq 1$,

$$|(1 - \hat{\varphi}_\varepsilon)\langle\xi\rangle^{s-t}| < \delta$$

everywhere. Plugging this bound into (1.1) shows that

$$||v - v * \varphi_\varepsilon||_{H^s(\mathbf{R}^d)} \leq \delta||\hat{v}\langle\xi\rangle^t|| = \delta||v||_{H^t(\mathbf{R}^d)},$$

which is what we wanted to show.

Finally, we show that u_{n_k} (as above) is Cauchy in $H^s(\mathbf{R}^d)$, which suffices to prove the compactness of the inclusion. Suppose M is a uniform upper bound on $||u_n||_{H^t(\mathbf{R}^d)}$. Fix $\delta > 0$, and use the previous paragraph to choose $m > 0$ large enough so that

$$||v * \varphi_{1/m} - v|_{H^s(\mathbf{R}^d)} \leq \frac{\delta}{3M}||v||_{H^t}$$

for all $v \in H^t(\mathbf{R}^d)$. Then

$$||u_{n_k} - u_{n_j}||_{H^s(\mathbf{R}^d)} \leq ||u_{n_k} - u_{n_k} * \varphi_{1/m}||_{H^s(\mathbf{R}^d)} + ||u_{n_j} - u_{n_j} * \varphi_{1/m}||_{H^s(\mathbf{R}^d)} + ||u_{n_k} * \varphi_{1/m} - u_{n_j} * \varphi_{1/m}||_{H^s(\mathbf{R}^d)}.$$

The first two terms are less than $\frac{1}{3}\delta$ each, and the last term can be made arbtrarily small for k, j large enough. This proves the Theorem.